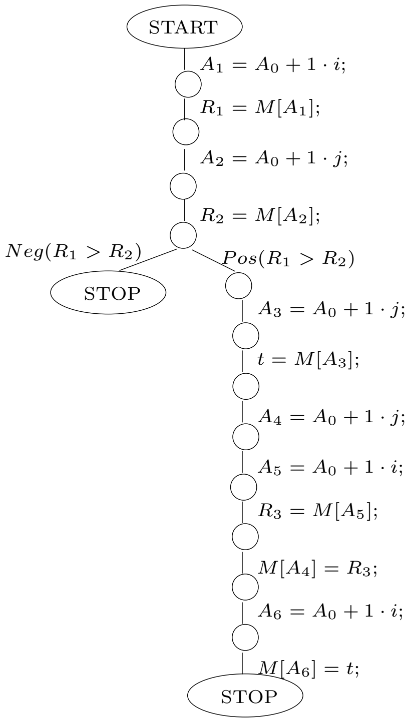
Program Optimisation
Solutions of Homework 1

1. Here is the control-flow graph of the function `swap`.



   a) The set of available expressions at each program point is indicated in Abbildung 1.

   b) Applying transformation 1 gives the control flow graph in Abbildung 2. Applying transformation 2 gives the control flow graph in Abbildung 3.

2. a) Let the lattice of Booleans be named $\mathbb{D} = \{0, 1\}$. We have the lattice $M = \{f_1, f_2, \ldots, f_6\}$ where $f_1(x, y) = 0$, $f_2(x, y) = x \wedge y$, $f_3(x, y) = x$, $f_4(x, y) = y$, $f_5(x, y) = x \vee y$ and $f_6(x, y) = 1$. Then the only possible monotone functions from M to $\mathbb{D}$ are as in the table below. I.e. $[M \to \mathbb{D}] = \{F_1, F_2, \ldots, F_8\}$.

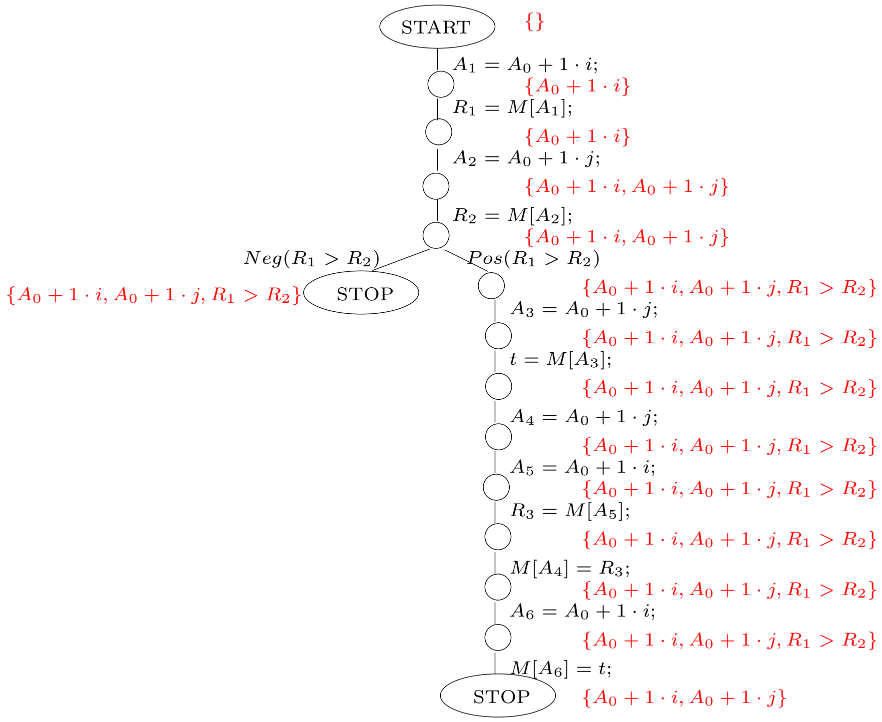


Abbildung 1: Available expressions

x	$F_1(x)$	$F_2(x)$	$F_3(x)$	$F_4(x)$	$F_5(x)$	$F_6(x)$	$F_7(x)$	$F_8(x)$
f_1	0	0	0	0	0	0	0	1
f_2	0	0	0	0	0	0	1	1
f_3	0	0	0	1	0	1	1	1
f_4	0	0	0	0	1	1	1	1
f_5	0	0	1	1	1	1	1	1
f_6	0	1	1	1	1	1	1	1

b) Their ordering is as shown in Abbildung 4

3. a) First we show that the properties of a partial order are satisfied.
   - **Reflexivity:** Let $x \in \mathbb{D}_1$ and $y \in \mathbb{D}_2$. Since $\mathbb{D}_1$ and $\mathbb{D}_2$ are partial orders we have $x \sqsubseteq x$ and $y \sqsubseteq y$. Hence $(x, y) \sqsubseteq (x, y)$.
   - **Anti-symmetry:** Let $(x_1, y_1), (x_2, y_2) \in \mathbb{D}_1 \times \mathbb{D}_2$ such that $(x_1, y_1) \sqsubseteq (x_2, y_2)$ and $(x_2, y_2) \sqsubseteq (x_1, y_1)$. Hence we have
     $x_1 \sqsubseteq x_2,$ (1)
     $y_1 \sqsubseteq y_2,$ (2)
     $x_2 \sqsubseteq x_1,$ (3)
     $y_2 \sqsubseteq y_1,$ (4)
     Since $\mathbb{D}_1$ is a partial order hence from (1) and (3) we have $x_1 = x_2$. Since $\mathbb{D}_2$ is a partial order hence from (2) and (4) we have $y_1 = y_2$. Hence we have $(x_1, y_1) = (x_2, y_2)$.

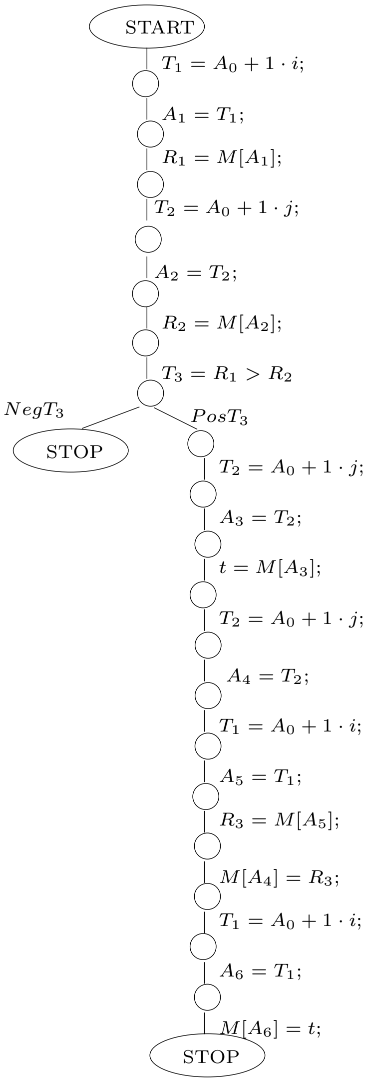


Abbildung 2: Application of transformation 1

- **Transitivity:** Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{D}_1 \times \mathbb{D}_2$ such that $(x_1, y_1) \sqsubseteq (x_2, y_2)$ and $(x_2, y_2) \sqsubseteq (x_3, y_3)$. Hence we have
  $x_1 \sqsubseteq x_2, \quad (1)$
  $y_1 \sqsubseteq y_2, \quad (2)$
  $x_2 \sqsubseteq x_3, \quad (3)$
  $y_2 \sqsubseteq y_3, \quad (4)$
  Since $\mathbb{D}_1$ is a partial order, hence from (1) and (3) we have $x_1 \sqsubseteq x_3$. Since $\mathbb{D}_2$ is a partial order hence from (2) and (4) we have $y_1 \sqsubseteq y_3$. Hence we have $(x_1, y_1) \sqsubseteq (x_3, y_3)$.

We have shown that $\mathbb{D}_1 \times \mathbb{D}_2$ is a partial order. Now let $X \subseteq \mathbb{D}_1 \times \mathbb{D}_2$. We have to show that X has a lub. Let $X_1 = \{x \mid (x, y) \in X\}$ and $X_2 = \{y \mid (x, y) \in X\}$. Since $\mathbb{D}_1$ is a complete lattice we have some $a_1 = \bigsqcup X_1$. Since $\mathbb{D}_2$ is a complete lattice we have some $a_2 = \bigsqcup X_2$.

- We first show that (a_1, a_2) is an upper bound of X. Let $(x, y) \in X$. Then $x \in X_1$. Since a_1 is an upper bound of X_1 we have $x \sqsubseteq a_1$. Similarly we

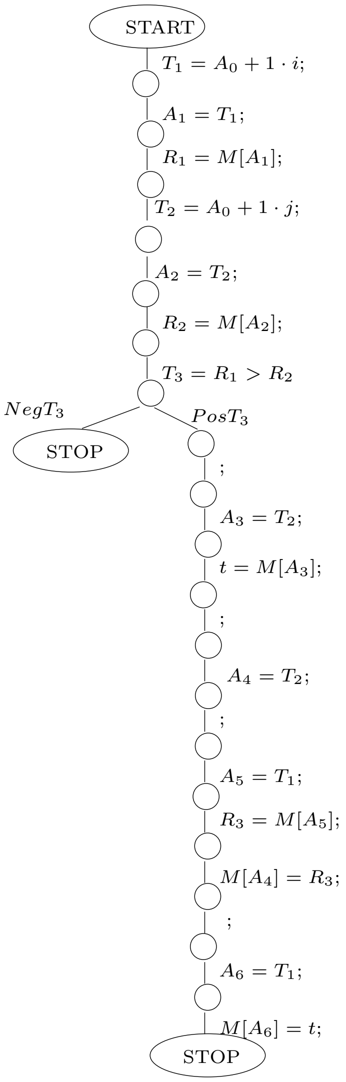


Abbildung 3: Application of transformation 2

have $y \sqsubseteq a_2$. Hence we have $(x, y) \sqsubseteq (a_1, a_2)$.

- Next let (b_1, b_2) be some upper bound of X. We have to show that $(a_1, a_2) \sqsubseteq (b_1, b_2)$.
  - First we show that that b_1 is an upper bound of X_1. Let $x \in X_1$. Then there is some y such that $(x, y) \in X$. Hence $(x, y) \sqsubseteq (b_1, b_2)$ because (b_1, b_2) is an upper bound of X. Hence $x \sqsubseteq b_1$.

  We have shown that b_1 is an upper bound of X_1. But a_1 is the lub of X_1 hence we must have $a_1 \sqsubseteq b_1$. Similarly we show that $a_2 \sqsubseteq b_2$. Hence we have $(a_1, a_2) \sqsubseteq (b_1, b_2)$.

Thus we have shown that $(a_1, a_2) = \bigsqcup X$. Hence $\mathbb{D}_1 \times \mathbb{D}_2$ is a complete lattice.

b) - **Part 1:** Assume that f is monotone. To show that f_x is monotone for $x \in \mathbb{D}_1$, we take any $y_1, y_2 \in \mathbb{D}_2$ such that $y_1 \sqsubseteq y_2$. By reflexivity of $\mathbb{D}_1$ we have $(x, y_1) \sqsubseteq (x, y_2)$. Since f is monotone, hence we have $f(x, y_1) \sqsubseteq f(x, y_2)$. Hence $f_x(y_1) \sqsubseteq f_x(y_2)$. Hence f_x is monotone. Similarly we show that f_y

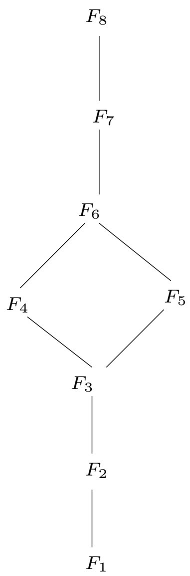


Abbildung 4: Ordering of elements of $[M \to \mathbb{D}]$

is monotone for $y \in \mathbb{D}_2$.

- **Part 2:** Assume that f_x and f_y are monotone for all $x \in \mathbb{D}_1, y \in \mathbb{D}_2$. To show that f is monotone take any $(x_1, y_1), (x_2, y_2) \in \mathbb{D}_1 \times \mathbb{D}_2$ such that $(x_1, y_1) \sqsubseteq (x_2, y_2)$. Then we have $x_1 \sqsubseteq x_2$ and $y_1 \sqsubseteq y_2$. Since f_{x_1} is monotone we have $f_{x_1}(y_1) \sqsubseteq f_{x_1}(y_2) = f_{y_2}(x_1)$. Since f_{y_2} is monotone we have $f_{y_2}(x_1) \sqsubseteq f_{y_2}(x_2)$. By transitivity of $\mathbb{D}$ we have $f_{x_1}(y_1) \sqsubseteq f_{y_2}(x_2)$. Hence $f(x_1, y_1) \sqsubseteq f(x_2, y_2)$.

4. a) We have
   $f^0(x) = x$
   $f^1(x) = (x \cap a) \cup b$
   $f^2(x) = ((x \cap a) \cup b) \cap a \cup b = (x \cap a \cap a) \cup (b \cap a) \cup b = (x \cap a) \cup (b \cap a) \cup b = (x \cap a) \cup b$
   As $f^2(x) = f^1(x)$ hence $f^i(x) = f^1(x)$ for all $i \geq 1$. Hence $f^*(x) = f^0(x) \sqcup f^1(x) = f^0(x) \cup f^1(x) = x \cup (x \cap a) \cup b = x \cup b$.

   b) We have $f^*(x) = \bigsqcup\{x, x+1, x+2, x+3, \ldots\} = \infty$.

   c) We have $f^i(0) = 0$ for all i. Hence $f^*(0) = \bigsqcup\{0, 0, 0, \ldots\} = 0$. For $x \geq 1$ we have $f^i(x) = 2^i x$. Hence $f^*(x) = \bigsqcup\{x, 2x, 4x, 8x, \ldots\} = \infty$. Thus

$$f^*(x) = \begin{cases} 0 & \text{if } x = 0 \\ \infty & \text{otherwise} \end{cases}$$