Program Optimisation
Solutions of Homework 3

1. a) The elements of the partial order are $\oplus$, $\ominus$ and $\top$. We intend $\oplus$ to describe positive values, and $\ominus$ to describe negative values. The ordering is: $\oplus \sqsubset \top$ and $\ominus \sqsubset \top$.

   b) The description relation Δ is defined as follows.

$$x \Delta \oplus \text{ iff } x \geq 0$$
$$x \Delta \ominus \text{ iff } x < 0$$
$$x \Delta \top \text{ iff } x \in \mathbb{Z}$$

   We have chosen $\oplus$ to describe positive values as well as 0, and $\ominus$ to describe strictly negative values. There are also other possibilities.

   c) The abstract operator $+^\sharp$ is defined as

$$a +^\sharp b = \begin{cases} \oplus & \text{if } a = \oplus \text{ and } b = \oplus \\ \ominus & \text{if } a = \ominus \text{ and } b = \ominus \\ \top & \text{otherwise} \end{cases}$$

   The abstract operator $-^\sharp$ is defined as

$$a -^\sharp b = \begin{cases} \oplus & \text{if } a = \oplus \text{ and } b = \ominus \\ \ominus & \text{if } a = \ominus \text{ and } b = \oplus \\ \top & \text{otherwise} \end{cases}$$

   We similarly define abstract operators corresponding to other concrete operators.

   d) We show that $+^\sharp$ respects Δ. The arguments for the other operators are similar. We need to show that if $x \Delta a$ and $y \Delta b$ then $x + y \Delta a +^\sharp b$. If $a = \oplus$ and $b = \oplus$ then $x \geq 0$ and $y \geq 0$ hence $x + y \geq 0$ so that $x + y \Delta \oplus = a +^\sharp b$. If $a = \ominus$ and $b = \ominus$ then $x < 0$ and $y < 0$ hence $x + y < 0$ so that $x + y \Delta \ominus = a +^\sharp b$. In all other cases, trivially we have $x + y \Delta \top = a +^\sharp b$.

   e) We consider edge transformations for the condition $Pos(e_1 == e_2)$. Other conditions are treated similarly.

$$[\![Pos(e_1 == e_2)]\!]^\sharp D = \begin{cases} \bot & \text{if } [\![e_1]\!]^\sharp D = \oplus \text{ and } [\![e_2]\!]^\sharp D = \ominus \\ \bot & \text{if } [\![e_1]\!]^\sharp D = \ominus \text{ and } [\![e_2]\!]^\sharp D = \oplus \\ D & \text{otherwise} \end{cases}$$

   To show its correctness we assume $(\rho, \mu) \Delta D$, $[\![Pos(e_1 == e_2)]\!](\rho, \mu) = (\rho_1, \mu_1)$, $[\![Pos(e_1 == e_2)]\!]^\sharp D = D_1$, and show that $(\rho_1, \mu_1) \Delta D_1$. Since

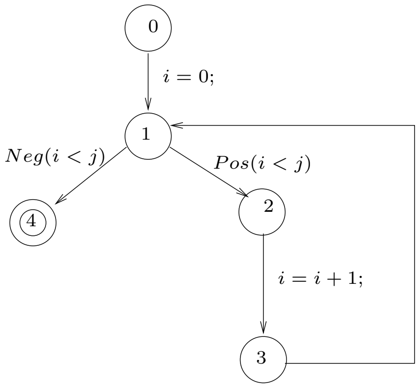

Abbildung 1: Program for which interval analysis without widening does not terminate

$[\![Pos(e_1 == e_2)]\!](\rho, \mu) = (\rho_1, \mu_1)$ it means that the condition $Pos(e_1 == e_2)$ succeeds for the state (ρ, μ), i.e. $[\![e_1]\!](\rho, \mu) = [\![e_2]\!](\rho, \mu)$. We have $(\rho_1, \mu_1) = (\rho, \mu)$. From above definition we must have $D_1 = D$. Hence $(\rho_1, \mu_1)\Delta D_1$.

2. The abstract multiplication for intervals is defined as $[l_1, u_1] *^\sharp [l_2, u_2] = [a, b]$ where $a = l_1l_2 \sqcap l_1u_2 \sqcap u_1l_2 \sqcap u_1u_2$ and $b = l_1l_2 \sqcup l_1u_2 \sqcup u_1l_2 \sqcup u_1u_2$. Now suppose $l_1 \leq z_1 \leq u_1$ and $l_2 \leq z_2 \leq u_2$. We have to show that $a \leq z_1z_2 \leq b$. We do case analysis on the ordering between the values $l_1, u_1, l_2, u_2, 0$ and show that in each case we have $a \leq z_1z_2 \leq b$.

3. We define

$$!^\sharp[l, u] = \begin{cases} [1, 1] & \text{if } l = 0 \text{ and } u = 0 \\ [0, 0] & \text{if } 0 < l \text{ or } u < 0 \\ [0, 1] & \text{otherwise} \end{cases}$$

$$[l_1, u_1] \neq^\sharp [l_2, u_2] = \begin{cases} [1, 1] & \text{if } u_1 < l_2 \text{ or } u_2 < l_1 \\ [0, 0] & \text{if } l_1 = u_1 = l_2 = u_2 \\ [0, 1] & \text{otherwise} \end{cases}$$

4. For the example program in Abbildung 1, interval analysis without widening proceeds as follows, and does not terminate:

	1				2				3				
	l_i	u_i	l_j	u_j	l_i	u_i	l_j	u_j	l_i	u_i	l_j	u_j	
0	$-\infty$	$+\infty$	$-\infty$	$+\infty$	1	1	$-\infty$	$+\infty$	2	2	$-\infty$	$+\infty$	
1	0	0	$-\infty$	$+\infty$	1	1	$-\infty$	$+\infty$	2	2	$-\infty$	$+\infty$	...
2	0	0	$-\infty$	$+\infty$	1	1	$-\infty$	$+\infty$	2	2	$-\infty$	$+\infty$	
3	1	1	$-\infty$	$+\infty$	2	2	$-\infty$	$+\infty$	3	3	$-\infty$	$+\infty$	