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Two kinds of semantics of programs:

- Static semantics: types
- Dynamic semantics: execution of the program

Type safety: "Well types programs never go wrong"

– Robin Milner

Standard methodology: Safety = Progress + Preservation

Progress: a well types program that is not a value can be evaluated further

Preservation: well typed programs remain so during evaluation.

A simple functional language (the simply typed lambda calculus)

t	$::=$		terms:
		x	variable
	$\mid$	0	
	$\mid$	$\mathsf{succ}\ t \mid \mathsf{pred}\ t$	
	$\mid$	$\mathsf{iszero}\ t$	zero test
	$\mid$	$\mathsf{true} \mid \mathsf{false}$	
	$\mid$	$\mathsf{if}\ t\ \mathsf{then}\ t\ \mathsf{else}\ t$	
	$\mid$	$\mathsf{fun}\ x : T \cdot t$	functions
	$\mid$	$\mathsf{apply}\ (t\ , t\ )$	application

The types

$T ::=$		
	Bool	type of Booleans
	Int	type of ints
	$T \longrightarrow T$	type of functions

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	Bool	type of Booleans
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The results of computations

v ::=		values:
	true | false	Boolean values
|	nv	numerical value
|	fun $x : T \cdot t$	functional value

nv ::=	
	0
|	succ nv

The Dynamic Semantics: Evaluation

$$\frac{t \longrightarrow t'}{\mathsf{succ}\ t \longrightarrow \mathsf{succ}\ t'} \text{ (E-Succ)}$$

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$$\mathsf{pred}\ 0 \longrightarrow 0 \text{ (E-PredZero)}$$

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$$\mathsf{pred}\ 0 \longrightarrow 0 \text{ (E-PredZero)} \qquad \mathsf{pred}\ (\mathsf{succ}\ nv) \longrightarrow nv \text{ (E-PredSucc)}$$

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$$\frac{t \longrightarrow t'}{\textsf{iszero}\ t \longrightarrow \textsf{iszero}\ t'} \text{ (E-IsZero)}$$

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$$\textsf{iszero}\ 0 \longrightarrow \textsf{true}\quad \text{(E-IsZeroZero)}$$

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$$\textsf{iszero}\ 0 \longrightarrow \textsf{true} \text{ (E-IsZeroZero)} \qquad \textsf{iszero}\ (\textsf{succ}\ nv) \longrightarrow \textsf{false} \text{ (E-IsZeroSucc)}$$

$$\frac{t \longrightarrow t'}{\textsf{if}\ t\ \textsf{then}\ t_1\ \textsf{else}\ t_2 \longrightarrow \textsf{if}\ t'\ \textsf{then}\ t_1\ \textsf{else}\ t_2} \text{ (E-If)}$$

$\text{if true then } t_1 \text{ else } t_2 \longrightarrow t_1$ (E-IfTrue)

$$\textsf{if true then } t_1 \textsf{ else } t_2 \longrightarrow t_1 \quad \text{(E-IfTrue)} \quad \textsf{if false then } t_1 \textsf{ else } t_2 \longrightarrow t_2 \quad \text{(E-IfFalse)}$$

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$$\frac{t_1 \longrightarrow t_1'}{\text{apply } (t_1, t_2) \longrightarrow \text{apply } (t_1', t_2)} \ \text{(E-App1)}$$

$$\text{if true then } t_1 \text{ else } t_2 \longrightarrow t_1 \quad \text{(E-IfTrue)}$$

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$$\frac{t_2 \longrightarrow t_2'}{\text{apply } (v_1, t_2) \longrightarrow \text{apply } (v_1, t_2')} \quad \text{(E-App2)}$$

$$\textsf{if true then } t_1 \textsf{ else } t_2 \longrightarrow t_1 \quad \text{(E-IfTrue)} \qquad \textsf{if false then } t_1 \textsf{ else } t_2 \longrightarrow t_2 \quad \text{(E-IfFalse)}$$

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$$\textsf{apply } (\textsf{fun } x : T \cdot t, v) \longrightarrow t\,[x \mapsto v] \text{ (E-App)}$$

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Substitutions are defined as usual.

$$(\text{if true then } (\text{pred } x) \text{ else } 0)\, [x \mapsto \text{succ } 0] = (\text{if true then } (\text{pred } (\text{succ } 0)) \text{ else } 0)$$

$$\text{if true then } t_1 \text{ else } t_2 \longrightarrow t_1 \quad \text{(E-IfTrue)} \qquad \text{if false then } t_1 \text{ else } t_2 \longrightarrow t_2 \quad \text{(E-IfFalse)}$$

$$\frac{t_1 \longrightarrow t_1'}{\text{apply } (t_1, t_2) \longrightarrow \text{apply } (t_1', t_2)} \text{ (E-App1)}$$

$$\frac{t_2 \longrightarrow t_2'}{\text{apply } (v_1, t_2) \longrightarrow \text{apply } (v_1, t_2')} \text{ (E-App2)}$$

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$$(\text{fun } x : \text{Int} \cdot \text{if true then } x \text{ else succ } (y))[y \mapsto \text{succ } (x)]$$
$$= (\text{fun } z : \text{Int} \cdot \text{if true then } z \text{ else succ } (\text{succ } (x)))$$

Example

$$\begin{array}{ll} & \mathsf{apply}\ (\mathsf{fun}\ x : \mathsf{Int} \cdot \mathsf{if}\ x\ \mathsf{then}\ (\mathsf{pred}\ (\mathsf{succ}\ 0))\ \mathsf{else}\ (\mathsf{succ}\ 0), \mathsf{iszero}\ 0) \\ \longrightarrow & \mathsf{apply}\ (\mathsf{fun}\ x : \mathsf{Int} \cdot \mathsf{if}\ x\ \mathsf{then}\ (\mathsf{pred}\ (\mathsf{succ}\ 0))\ \mathsf{else}\ (\mathsf{succ}\ 0), \mathsf{true}\ ) \\ \longrightarrow & \mathsf{if}\ \mathsf{true}\ \mathsf{then}\ (\mathsf{pred}\ (\mathsf{succ}\ 0))\ \mathsf{else}\ (\mathsf{succ}\ 0) \\ \longrightarrow & (\mathsf{pred}\ (\mathsf{succ}\ 0)) \\ \longrightarrow & 0 \end{array}$$

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The justification for the first evaluation step is as follows

$$\dfrac{\dfrac{}{\textsf{iszero}\ 0 \longrightarrow \textsf{true}}\ \text{(E-IsZeroZero)}}{\textsf{apply}\ (\textsf{fun}\ x : \textsf{Int} \cdot \textsf{if}\ \ \ldots, \textsf{iszero}\ 0) \longrightarrow \textsf{apply}\ (\textsf{fun}\ x : \textsf{Int} \cdot \textsf{if}\ \ \ldots, \textsf{true}\ )}\ \text{(E-App2)}$$

A program which gets stuck during evaluation

$$\begin{array}{ll} & \mathsf{apply}\ (\mathsf{fun}\ x : \mathsf{Int} \cdot \mathsf{if}\ x\ \mathsf{then}\ (\mathsf{pred}\ (\mathsf{succ}\ 0))\ \mathsf{else}\ (\mathsf{succ}\ 0), 0) \\ \longrightarrow & \mathsf{if}\ 0\ \mathsf{then}\ (\mathsf{pred}\ (\mathsf{succ}\ 0))\ \mathsf{else}\ (\mathsf{succ}\ 0), \end{array}$$

There are no rules for evaluating this program further.

This program is not yet a value.

The type system of a type-safe language should reject such programs.

The Static Semantics: Typing

A type environment Γ is of the form $x_1 : T_1, \ldots, x_n : T_n$

$$\frac{x : T \in \Gamma}{\Gamma \vdash x : T} \text{ (T-Var)}$$

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$$\Gamma \vdash \mathsf{true} : \mathsf{Bool} \text{ (T-True)}$$

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$$\frac{\Gamma \vdash t \ : \mathsf{Int}}{\Gamma \vdash \mathsf{iszero}\ t \ : \mathsf{Bool}} \text{(T-IsZero)}$$

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$$\Gamma \vdash \mathsf{true} : \mathsf{Bool} \text{ (T-True)}$$

$$\Gamma \vdash \mathsf{false} : \mathsf{Bool} \text{ (T-False)}$$

$$\frac{\Gamma \vdash t : \mathsf{Int}}{\Gamma \vdash \mathsf{iszero}\ t : \mathsf{Bool}} \text{ (T-IsZero)}$$

$$\frac{\Gamma \vdash t : \mathsf{Bool} \quad \Gamma \vdash t_1 : T \quad \Gamma \vdash t_2 : T}{\Gamma \vdash \mathsf{if}\ t\ \mathsf{then}\ t_1\ \mathsf{else}\ t_2 : T} \text{ (T-If)}$$

$$\frac{\Gamma, x : T \vdash t \; : T'}{\Gamma \vdash \textsf{fun}\ x : T \cdot t \; : T \to T'} \text{(T-Fun)}$$

$$\frac{\Gamma, x : T \vdash t \ : T'}{\Gamma \vdash \mathsf{fun}\ x : T \cdot t \ : T \longrightarrow T'} \text{ (T-Fun)}$$

$$\frac{\Gamma \vdash t_1 \ : T \longrightarrow T' \quad \Gamma \vdash t_2 \ : T}{\Gamma \vdash \mathsf{apply}\ (t_1 \ , t_2 \ ) : T'} \text{ (T-App)}$$

Example

$$
\dfrac{\dfrac{}{\vdots}\quad
\vdash \mathsf{fun}\ x : \mathsf{Bool} \cdot \mathsf{if}\ x\ \mathsf{then}\ (\mathsf{pred}\ 0)\ \mathsf{else}\ (\mathsf{succ}\ 0) : \mathsf{Bool} \to \mathsf{Int}
\qquad
\dfrac{\dfrac{}{\vdash 0 : \mathsf{Int}}\ (\text{T-Zero})}{\vdash \mathsf{iszero}\ 0 : \mathsf{Bool}}\ (\text{T-IsZero})}
{\vdash \mathsf{apply}\ (\mathsf{fun}\ x : \mathsf{Bool} \cdot \mathsf{if}\ x\ \mathsf{then}\ (\mathsf{pred}\ 0)\ \mathsf{else}\ (\mathsf{succ}\ 0), \mathsf{iszero}\ 0) : \mathsf{Int}}\ (\text{T-App})
$$

Example

$$\dfrac{\dfrac{}{x : \mathsf{Bool} \vdash x : \mathsf{Bool}}\,(\text{T-Var}) \quad \dfrac{x : \mathsf{Bool} \vdash 0 : \mathsf{Int}}{x : \mathsf{Bool} \vdash (\mathsf{pred}\ 0) : \mathsf{Int}}\,(\text{T-Pred}) \quad \dfrac{x : \mathsf{Bool} \vdash 0 : \mathsf{Int}}{x : \mathsf{Bool} \vdash \mathsf{succ}\ 0 : \mathsf{Int}}\,(\text{T-Succ})}{\dfrac{x : \mathsf{Bool} \vdash \mathsf{if}\ x\ \mathsf{then}\ (\mathsf{pred}\ 0)\ \mathsf{else}\ (\mathsf{succ}\ 0) : \mathsf{Int}}{\vdash \mathsf{fun}\ x\ : \mathsf{Bool} \cdot \mathsf{if}\ x\ \mathsf{then}\ (\mathsf{pred}\ 0)\ \mathsf{else}\ (\mathsf{succ}\ 0) : \mathsf{Bool} \to \mathsf{Int}}\,(\text{T-Fun})}\,(\text{T-If})$$

$$\dfrac{\dfrac{\vdots}{\vdash \mathsf{fun}\ x\ : \mathsf{Bool} \cdot \mathsf{if}\ x\ \mathsf{then}\ (\mathsf{pred}\ 0)\ \mathsf{else}\ (\mathsf{succ}\ 0) : \mathsf{Bool} \to \mathsf{Int}} \qquad \dfrac{\dfrac{}{\vdash 0 : \mathsf{Int}}\,(\text{T-Zero})}{\vdash \mathsf{iszero}\ 0 : \mathsf{Bool}}\,(\text{T-IsZero})}{\vdash \mathsf{apply}\ (\mathsf{fun}\ x : \mathsf{Bool} \cdot \mathsf{if}\ x\ \mathsf{then}\ (\mathsf{pred}\ 0)\ \mathsf{else}\ (\mathsf{succ}\ 0), \mathsf{iszero}\ 0) : \mathsf{Int}}\,(\text{T-App})$$

The following program

$$\mathsf{if\ true\ then}\ (\mathsf{succ}\ 0)\ \mathsf{else}\ (\mathsf{iszero}\ 0)$$

evaluates to ($\mathsf{succ}\ 0$) (doesn't get stuck).

However it is not well-typed according to our type system, i.e. we cannot show

$$\vdash \mathsf{if\ true\ then}\ (\mathsf{succ}\ 0)\ \mathsf{else}\ (\mathsf{iszero}\ 0) : T$$

for any type T.

$\Longrightarrow$ we reject some safe programs.

The only required property for type safety is that all unsafe programs should be rejected.

The standard method for showing type safety.

(1) Progress

If $\vdash t : T$ and t is not a value then $t \longrightarrow t'$ for some term t'.

Well typed programs so not get stuck in some undefined state.

(2) Preservation

If $\vdash t : T$ and $t \longrightarrow t'$ then $\vdash t' : T$.

Evaluation preserves well-typedness (and type) of a program.

The proofs are usually easy (but long) once the right definitions have been found out.

Examples of type-safe languages: Java, SML.

Examples of type-unsafe languages: C, C++.

Progress: If $\vdash t : T$ and t is not a value then $t \longrightarrow t'$ for some term t'

Progress: If $\vdash t : T$ and t is not a value then $t \longrightarrow t'$ for some term t'

Proof: We do induction on the size of typing derivations.

– If t is true , false , 0 or fun $x : T \cdot t'$ then there is nothing to prove because these are values.

Progress: If $\vdash t : T$ and t is not a value then $t \longrightarrow t'$ for some term t'

Proof: We do induction on the size of typing derivations.

– If t is true , false , $\mathsf{0}$ or $\mathsf{fun}\ x : T \cdot t'$ then there is nothing to prove because these are values.

– t cannot be a variable because the only rule for typing a variable is

$$\frac{x : T \in \Gamma}{\Gamma \vdash x : T}\ \text{(T-Var)}$$

which requires Γ to be non-empty.

some intersting cases:

– If t is of the form $\mathsf{succ}\ t'$, the typing derivation must be

$$\frac{\vdash t' : \mathsf{Int}}{\vdash \mathsf{succ}\ t' : \mathsf{Int}}\ \text{(T-Succ)}$$

If t' is a value then t is also a value. Othwerwise by induction hypothesis we have

$$\frac{t' \longrightarrow t''}{\mathsf{succ}\ t' \longrightarrow \mathsf{succ}\ t''}\ \text{(E-Succ)}$$

– If t is of the form $\mathsf{pred}\ t'$ then the typing derivation must be

$$\frac{\vdash t' : \mathsf{Int}}{\vdash \mathsf{pred}\ t' : \mathsf{Int}}\ \text{(T-Pred)}$$

(1) If t' is value 0 then by (E-PredZero) we know that $\mathsf{pred}\ t' \longrightarrow 0$.

(2) If t' is value $\mathsf{succ}\ nv$ then by (E-PredSucc) we know that $\mathsf{pred}\ t' \longrightarrow nv$.

(3) Otherwise t' is not a value. Hence by induction hypothesis we have

$$\frac{t' \longrightarrow t''}{\mathsf{pred}\ t' \longrightarrow \mathsf{pred}\ t''}\ \text{(E-Pred)}$$

– If t is of the form $\mathsf{iszero}\ t'$ then the typing derivation must be

$$\frac{\vdash t : \mathsf{Int}}{\vdash \mathsf{iszero}\ t : \mathsf{Bool}}\ \text{(T-IsZero)}$$

(1) If t' is value $\mathsf{0}$ then by (E-IsZeroZero) we know that $\mathsf{iszero}\ t' \longrightarrow \mathsf{true}$.

(2) If t' is value $\mathsf{succ}\ nv$ then by (E-IsZeroSucc) we know that $\mathsf{iszero}\ t' \longrightarrow \mathsf{false}$

(3) Otherwise t' is not a value and by induction hypothesis we have

$$\frac{t' \longrightarrow t''}{\mathsf{iszero}\ t' \longrightarrow \mathsf{iszero}\ t''}\ \text{(E-IsZero)}$$

– If t is of the form $\textsf{if } t_1 \textsf{ then } t_2 \textsf{ else } t_3$ then the typing derivation must be

$$\frac{\vdash t_1 : \textsf{Bool} \qquad \vdash t_2 : T \qquad \vdash t_3 : T}{\vdash \textsf{if } t_1 \textsf{ then } t_2 \textsf{ else } t_3 : T} \text{(T-If)}$$

(1) If t_1 is value $\textsf{true}$ then by (E-IfTrue) we know that $t \longrightarrow t_2$.

(2) If t_1 is value $\textsf{false}$ then by (E-IfFalse) we know that $t \longrightarrow t_3$.

(3) Otherwise t_1 is not a value and by induction hypothesis we have

$$\frac{t_1 \longrightarrow t_1'}{\textsf{if } t_1 \textsf{ then } t_2 \textsf{ else } t_3 \longrightarrow \textsf{if } t_1' \textsf{ then } t_2 \textsf{ else } t_3} \text{(E-If)}$$

– If t is of the form $\mathsf{apply}\ (t_1, t_2)$ then the typing derivation must be

$$\frac{\vdash t_1 : T \rightarrow T' \qquad \vdash t_2 : T}{\vdash \mathsf{apply}\ (t_1, t_2) : T'}\ \text{(T-App)}$$

(1) If t_1 is not a value then by induction hypothesis we have

$$\frac{t_1 \longrightarrow t_1'}{\mathsf{apply}\ (t_1, t_2) \longrightarrow \mathsf{apply}\ (t_1', t_2)}\ \text{(E-App1)}$$

(2) If t_1 is value v_1 and t_2 is not a value then by induction hypothesis we have

$$\frac{t_2 \longrightarrow t_2'}{\mathsf{apply}\ (v_1, t_2) \longrightarrow \mathsf{apply}\ (v_1, t_2')}\ \text{(E-App2)}$$

(3) Suppose t_1 is a value and t_2 is also a value v_2 . Since $\vdash t_1 : T \rightarrow T'$ the value t_1 must be $\mathsf{fun}\ x : T \cdot t_1'$. Hence by (E-App) we have

$$\mathsf{apply}\ (\mathsf{fun}\ x : T \cdot t_1' , v_2) \longrightarrow t_1' [x \mapsto v_2]$$

:-)

Preservation: If $\vdash t : T$ and $t \longrightarrow t'$ then $\vdash t' : T$

Preservation: If $\vdash t : T$ and $t \longrightarrow t'$ then $\vdash t' : T$

Proof: induction on typing derivations.

Some interesting cases:

– t is of the form $\textsf{if } t_1 \textsf{ then } t_2 \textsf{ else } t_3$. The typing derivation is of the form

$$\frac{\vdash t_1 : \textsf{Bool} \qquad \vdash t_2 : T \qquad \vdash t_3 : T}{\vdash \textsf{if } t_1 \textsf{ then } t_2 \textsf{ else } t_3 : T} \ (\text{T-If})$$

(1) Suppose $t_1 \longrightarrow t_1'$ so that $t \longrightarrow t'$ where t' is $\textsf{if } t_1' \textsf{ then } t_2 \textsf{ else } t_3$.

By induction hypothesis we know that $\Gamma \vdash t_1' : \textsf{Bool}$ so that $\Gamma \vdash t' : T$.

(2) Suppose t_1 is $\textsf{true}$ so that $t \longrightarrow t_2$ then we know that $\Gamma \vdash t_2 : T$.

– t is $\mathsf{apply}\ (\mathsf{fun}\ x : T' \cdot t_1\ , v_2)$ and the typing derivation is

$$\dfrac{\dfrac{x : T' \vdash t_1\ : T}{\vdash \mathsf{fun}\ x : T' \cdot t_1\ : T' \to T}\ (\text{T-Fun}) \qquad \vdash v_2 : T'}{\vdash \mathsf{apply}\ (\mathsf{fun}\ x : T' \cdot t_1\ , v_2) : T}\ (\text{T-App})$$

We have $t \longrightarrow t'$ where t' is $t_1\ [x \mapsto v_2]$.

To show that $\vdash t'\ : T$ we prove

Preservation of types under substitution

If $\Gamma, x : T' \vdash t_1\ : T$ and $\Gamma \vdash t_2\ : T'$ then $\Gamma \vdash t_1\ [x \mapsto t_2\ ] : T$.

Suppose now we extend the language by adding vectors.

$$
\begin{array}{rcll}
t & ::= & x \mid 0 & \\
& \mid & \ldots & \\
& \mid & [t\ , \ldots , t\ ] & \text{a vector of terms} \\
& \mid & \mathsf{get}\ t\ t & \text{accessing some ith element of a vector}
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New evaluation rules

$$\frac{t_i \longrightarrow t_i'}{[v_0,\ldots,v_{i-1},t_i\ ,t_{i+1}\ ,\ldots,t_n\ ] \longrightarrow [v_0,\ldots,v_{i-1},t_i'\ ,t_{i+1}\ ,\ldots,t_n\ ]} \text{ (E-Vec)}$$

$$\frac{t_1 \longrightarrow t_1'}{\mathsf{get}\ t_1\ t_2 \longrightarrow \mathsf{get}\ t_1'\ t_2} \text{ (E-Get1)}$$

$$\frac{t_2 \longrightarrow t_2'}{\mathsf{get}\ v_1\ t_2 \longrightarrow \mathsf{get}\ v_1\ t_2'} \text{ (E-Get2)}$$

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$$\frac{i \leq n}{\mathsf{get}\ \mathsf{succ}^{\,i}(0)\ [v_0, \ldots, v_n] \longrightarrow v_i} \text{ (E-Get)}$$

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New typing tules

$$\frac{\Gamma \vdash t_0 : T \ldots \Gamma \vdash t_n : T}{\Gamma \vdash [t_0, \ldots, t_n] : \mathsf{vector}\ T} \text{ (T-Vec)}$$

$$\frac{\Gamma \vdash t_1 : \mathsf{Int} \quad \Gamma \vdash t_2 : \mathsf{vector}\ T}{\Gamma \vdash \mathsf{get}\ t_1\ t_2 : T} \text{ (T-Get)}$$

Is the extended language type safe?

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No.

Preservation still holds, but progress fails.

Let term t be $\mathsf{get}\ (\mathsf{succ}\ (\mathsf{succ}\ (\mathsf{succ}\ 0)))[0, 0]$. It is well-typed.

$$\dfrac{\dfrac{\vdots}{\vdash (\mathsf{succ}\ (\mathsf{succ}\ (\mathsf{succ}\ 0))) : \mathsf{Int}} \quad \dfrac{\dfrac{}{\vdash 0 : \mathsf{Int}}\ (\text{T-Zero}) \quad \dfrac{}{\vdash 0 : \mathsf{Int}}\ (\text{T-Zero})}{\vdash [0, 0] : \mathsf{vector\ Int}}\ (\text{T-Vec})}{\vdash \mathsf{get}\ (\mathsf{succ}\ (\mathsf{succ}\ (\mathsf{succ}\ 0)))[0, 0] : \mathsf{Int}}\ (\text{T-Get})$$

But there is no term t' such that $t \longrightarrow t'$.

Remedy 1: Modify the typing rules to reject such programs.

Problem: type inference involves problems like precise array bounds checking at compile time.

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We introduce a new term for ill-defined states.

$$t \ ::= \ldots \mid \mathit{error}$$

and a rule for producing error message

$$\frac{i > n}{\textsf{get succ}\ ^{i}(0)\ [v_0, \ldots, v_n] \longrightarrow \mathit{error}}$$

and rules for propagating error messages

$$\mathsf{apply}\ (error, t\ ) \longrightarrow error \qquad \mathsf{apply}\ (v, error) \longrightarrow error\ \ldots$$

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$\mathsf{apply}\ (\mathit{error}, t\ ) \longrightarrow \mathit{error} \qquad \mathsf{apply}\ (v, \mathit{error}) \longrightarrow \mathit{error}\ \ldots$

Then we can show

Progress:

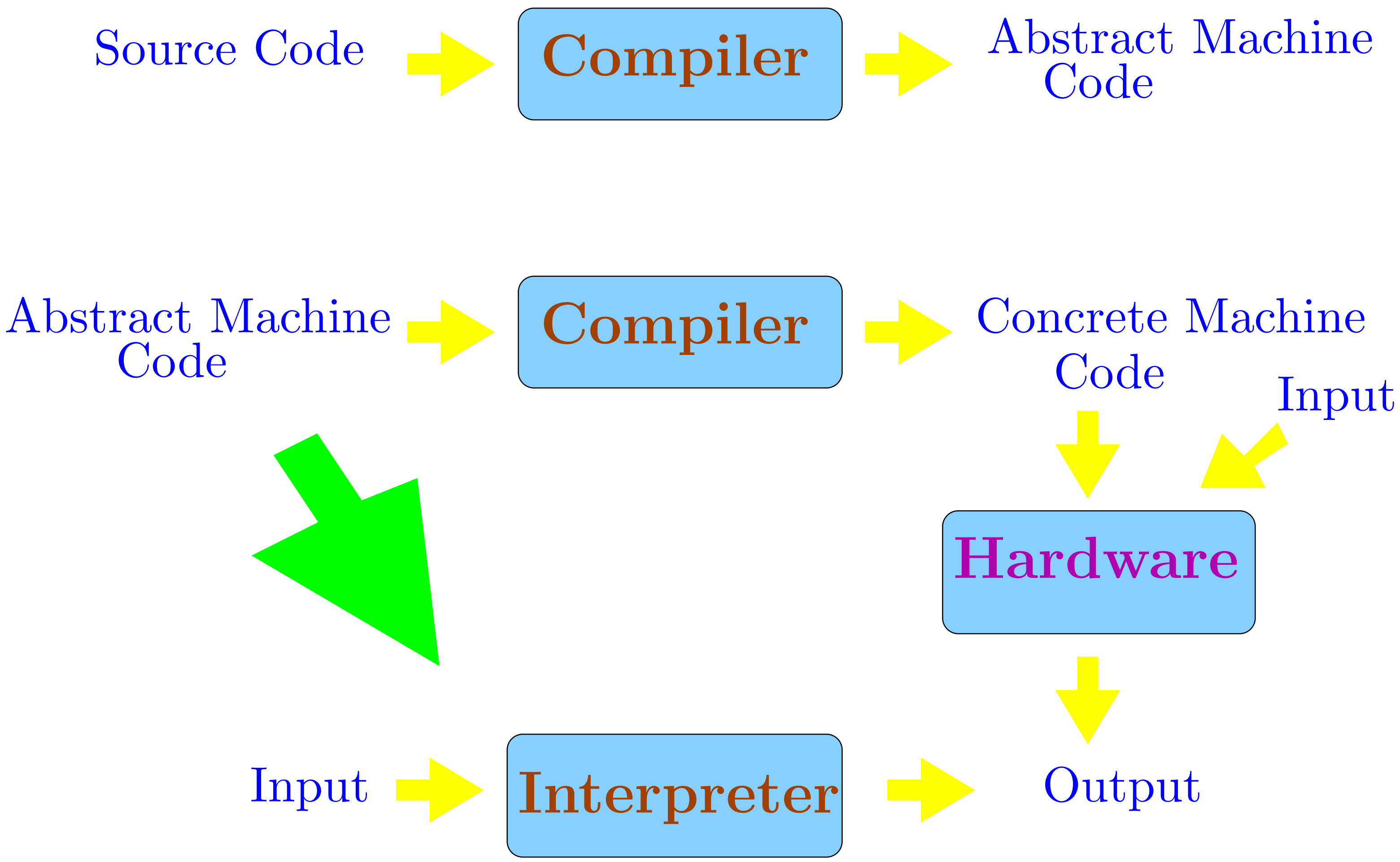
If $\vdash t\ : T$, t is not a value and $t \neq \mathit{error}$ then $t \longrightarrow t'$ for some t' .

Preservation:

If $\vdash t\ : T$ and $t \longrightarrow t'$ then either t' is error or $\vdash t'\ : T$.

Java Security

The virtual machine principle:



Java programs: definitions of classes.

```
public class hello {
   public static void main (String args[]) {
      System.out.println ("Hello!"); } }
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Compilation produces class files containing Java bytecode.

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javac hello.java
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produces file hello.class containing bytecodes for the class hello.

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$\Longrightarrow$ Portability

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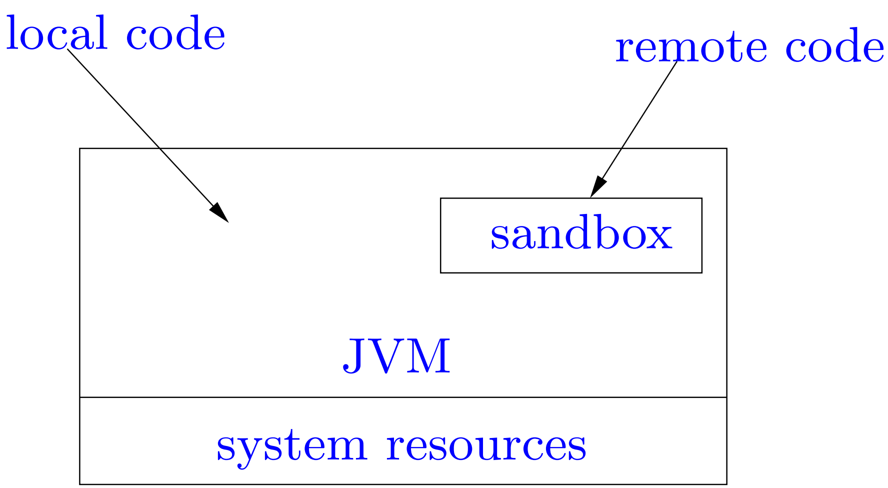

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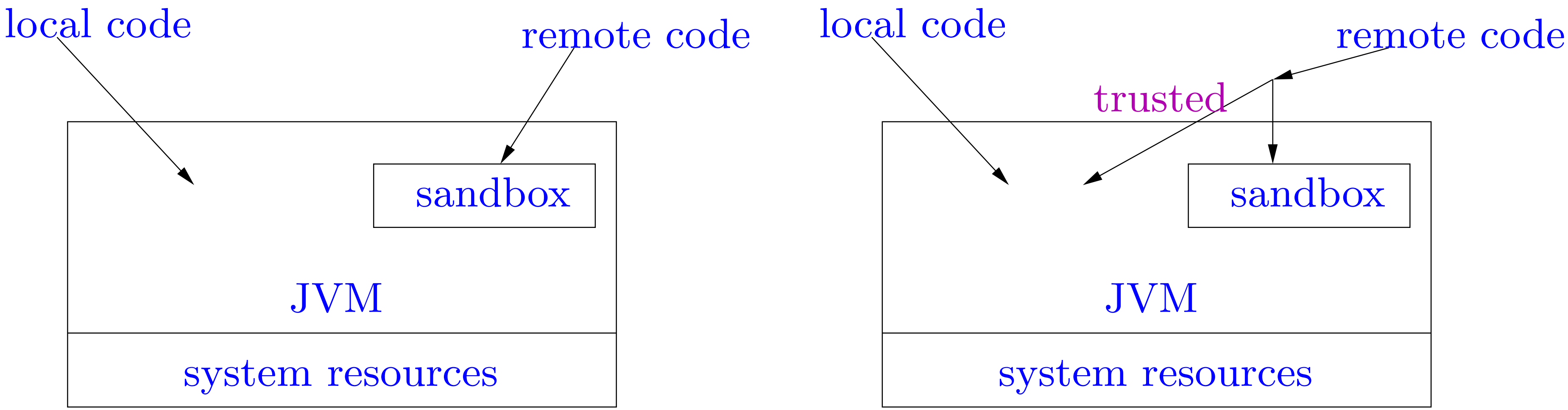
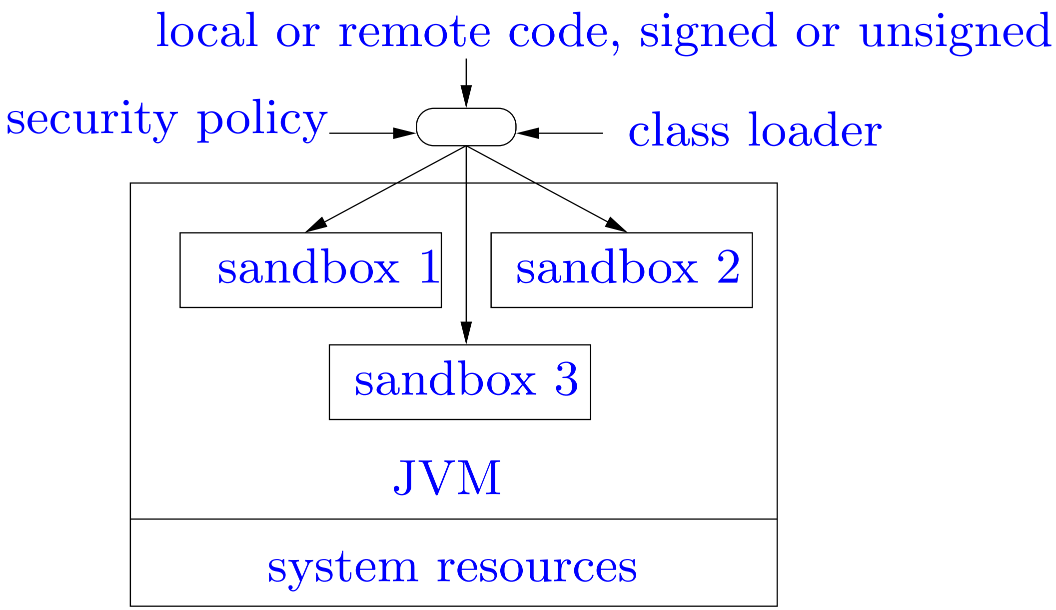
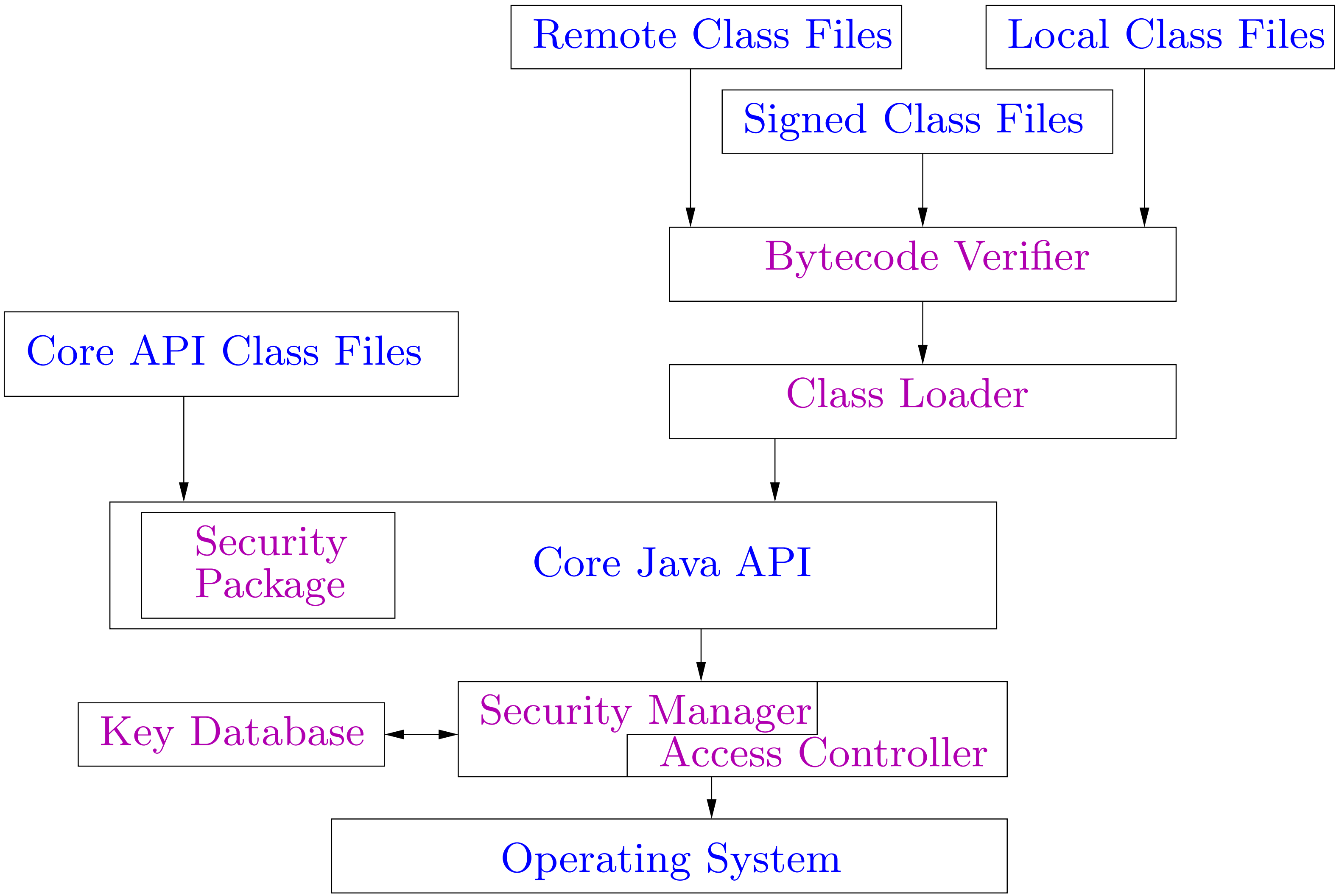


The current sandbox model in Java 2



Elements of the Java sandbox



Java language security constructs

Each entity has an access level

Specifier	Class	Package	Subclass	World
private	Yes	No	No	No
(Default)	Yes	Yes	No	No
protected	Yes	Yes	Yes	No
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Not sufficient for memory integrity . . .

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- No use of variables before initialization.
- Array bounds checks.
- No arbitrary casts between different classes.

```
public class A {private int x;}
public class B {public int x;}
...
// a is of class A
B b = (B) a;
// The above is rejected by the compiler
Object o = b; B b' = o;
// The above is allowed by compiler but raises exception at runtime
```

Enforcement of the Java language rules.

- At compile time:

check typing rules, enforcement of access qualifiers, prevention of most illegal type casts.

- At load time:

verify bytecodes when a class is loaded (prevent malicious bytecodes)

- At runtime:

raise exceptions for illegal type casts, out of bound array accesses, . . .